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VIBRATION OF A CIRCULAR CRACK UNDER THREE-DIMENSIONAL LOADING*

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The three-dimensional problem of harmonic oscillations of an elastic layer $x, y \in (-\infty, \infty)$, $-h \leqslant z \leqslant 0$ of thickness h, induced by the vibration of the upper and lower edges of circular crack in the plane $z = -c$ $(0 < c < h)$ parallel to the layer boundaries and occupying region Ω $(x^2 + y^2 \leqslant a^2)$. The crack edges are loaded by an arbitrary three-dimensional force system oscillating at frequency ω. The upper and lower boundaries of the layer are free of stresses. The respective boundary value problem is reduced to the solution of a system of three integral equations of the first kind, which is then transformed into a system of two integral equations and one independent separate equation, which are regularized by the method of factorization of functions and matrix-functions. As the result, the problem is reduced to a system of two integral equations and one separate integral equation which are of the second kind of the Fredholm type. These are then reduced to a finite-dimensional algebraic system.

A particular case of this problem was considered in /1/, where the vibration of a semi-infinite crack in an elastic medium was investigated. Note that in the problem formulation in /1/, the kernel matrix-function is considerably simplified so that the problem of matrix factorization does not arise, and the formulation of conditions of radiation is much simpler.

1. The input boundary value problem defined by three Lamé differential equations with related boundary conditions is reduced by the method of integral transforms with allowance for radiation into infinity /2,3/ to the solution of a system of three integral equations in the unknown vector $u^*(x, y)$ of the difference of the crack upper and lower edge displacements. In dimensionless matrix form this system can be defined as follows:

$$\iint_{\Omega} \mathbf{k}^*(x-\xi, y-\eta)\, \mathbf{u}^*(\xi, \eta)\, d\xi\, d\eta = \mathbf{f}^*(x, y), \quad x, y \in \Omega\ (x^2+y^2 \leqslant 1) \tag{1.1}$$

$$\mathbf{k}^*(x-\xi, y-\eta) = \frac{1}{4\pi^2} \int_{\Gamma_1}\int_{\Gamma_2} \mathbf{K}^*(\alpha, \beta)\, e^{-i[\alpha(x-\xi)+\beta(y-\eta)]}\, d\alpha\, d\beta$$

$$\mathbf{f}^*(x, y) = \frac{1}{4\pi^2} \int_{\Gamma_1}\int_{\Gamma_2} \mathbf{F}^*(\alpha, \beta)\, e^{-i(\alpha x+\beta y)}\, d\alpha\, d\beta$$

$$\mathbf{F}^*(\alpha, \beta) = \frac{a}{G}\, [\mathbf{T}^*(\alpha, \beta) + \mathbf{M}^*(\alpha, \beta)\, \mathbf{Q}^*(\alpha, \beta)] \tag{1.2}$$

Vectors $\mathbf{T}^*(\alpha, \beta)$ and $\mathbf{Q}^*(\alpha, \beta)$ are two-dimensional Fourier transforms in x and y of vectors $\boldsymbol{\tau}(x, y)$ and $\mathbf{q}^*(x, y)$ which are, respectively, the vectors of stress at the crack upper edge and of stress difference at its upper and lower edges. The selection of contours Γ_1 and Γ_2 conforms to data in /3,4/, and $\mathbf{K}^*(\alpha, \beta)$ and $\mathbf{M}^*(\alpha, \beta)$ are third order matrices whose elements, regular in the region containing contours Γ_1 and Γ_2, are of the form

$$\begin{aligned}
&K_{11}^*(\alpha, \beta) = \alpha^2 S + \beta^2 T, \quad K_{12}^*(\alpha, \beta) = K_{21}^*(\alpha, \beta) = \alpha\beta\, (S - T) \\
&K_{22}^*(\alpha, \beta) = \alpha^2 T + \beta^2 S, \quad K_{13}^*(\alpha, \beta) = -K_{31}^*(\alpha, \beta) = -i\alpha L \\
&K_{33}^*(\alpha, \beta) = K, \quad K_{23}^*(\alpha, \beta) = -K_{32}^*(\alpha, \beta) = -i\beta L \\
&M_{11}^*(\alpha, \beta) = \alpha^2 s + \beta^2 t, \quad M_{12}^*(\alpha, \beta) = M_{21}^*(\alpha, \beta) = \alpha\beta\, (s - t) \\
&M_{22}^*(\alpha, \beta) = \alpha^2 t + \beta^2 s, \quad M_{13}^*(\alpha, \beta) = i\alpha l, \quad M_{31}^*(\alpha, \beta) = i\alpha l_1 \\
&M_{33}^*(\alpha, \beta) = k, \quad M_{23}^*(\alpha, \beta) = i\beta l, \quad M_{32}^*(\alpha, \beta) = i\beta l_1
\end{aligned} \tag{1.3}$$

where

$$S = \frac{R}{Du^2}, \quad K = \frac{M}{D}, \quad T = -\frac{1}{Nu^2}, \quad L = \frac{P}{Du} \tag{1.4}$$

$$s = \frac{mR - pP}{Du^2}, \quad k = \frac{rM - pP}{D}, \quad t = -\frac{n}{Nu^2}, \quad l = \frac{pR - rP}{Du}$$

$$l_1 = \frac{mP - pM}{Du}, \quad D = P^2 - RM, \quad u^2 = \alpha^2 + \beta^2$$

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$$P = p(C) - p(H-C),\ R = r(C) + r(H-C)$$
$$M = m(C) + m(H-C),\ N = n(C) + n(H-C)$$
$$p = p(H-C),\ r = r(H-C),\ m = m(H-C),\ n = n(H-C)$$
$$C = \frac{c}{a},\quad H = \frac{h}{a},\quad \varkappa_2^2 = \frac{\rho\omega^2 a^2}{G},\quad \varkappa_1^2 = \frac{1-2\nu}{2(1-\nu)}\varkappa_2^2$$
$$\gamma_i^2 = u^2 - \varkappa_i^2,\ i = 1,2$$

$$p(d) = \frac{u}{\delta(d)}[2\gamma_1\gamma_2(\gamma_2^2+u^2)(\gamma_2^2+3u^2)(1-\operatorname{ch}(\gamma_1 d)\operatorname{ch}(\gamma_2 d)) + (8\gamma_1^2\gamma_2^2u^2 + (\gamma_2^2+u^2)^3)\operatorname{sh}(\gamma_1 d)\operatorname{sh}(\gamma_2 d)]$$

$$r(d) = \frac{\gamma_1\varkappa_2^2}{\delta(d)}[4\gamma_1\gamma_2 u^2 \operatorname{sh}(\gamma_1 d)\operatorname{ch}(\gamma_2 d) - (\gamma_2^2+u^2)^2\operatorname{ch}(\gamma_1 d)\operatorname{sh}(\gamma_2 d)]$$
$$m(d) = \frac{\gamma_2\varkappa_2^2}{\delta(d)}[4\gamma_1\gamma_2 u^2 \operatorname{ch}(\gamma_1 d)\operatorname{sh}(\gamma_2 d) - (\gamma_2^2+u^2)^2\operatorname{sh}(\gamma_1 d)\operatorname{ch}(\gamma_2 d)]$$
$$n(d) = \frac{1}{\gamma_2}\cdot\frac{\operatorname{ch}(\gamma_2 d)}{\operatorname{sh}(\gamma_2 d)}$$

$$\delta(d) = [(\gamma_2^2+u^2)^4 + 16\gamma_1^2\gamma_2^2u^4]\operatorname{sh}(\gamma_1 d)\operatorname{sh}(\gamma_2 d) + 8(\gamma_2^2+u^2)^2u^2\gamma_1\gamma_2[1-\operatorname{ch}(\gamma_1 d)\operatorname{ch}(\gamma_2 d)]$$

(a is the crack radius, and ρ, G, and ν are, respectively, the density, shear modulus and Poisson's coefficient of the material of the layer). If the crack is at the interface of two layers of different densities and elastic properties, it is necessary to consider in formulas (1.4) $p(C), r(C), m(C), n(C)$ as dependent on parameters of the upper layer of thickness c, and $p(H-C),\ r(H-C),\ m(H-C),\ n(H-C)$ on those of the lower layer of thickness $h-c$.

As $|u| \to \infty$ $(u^2 = \alpha^2+\beta^2)$, the asymptotic behavior of elements of matrices $\mathbf{K}^*(\alpha,\beta)$ and $\mathbf{M}^*(\alpha,\beta)$ is defined as follows:

$$\begin{aligned}&K_{ij}^*(\alpha,\beta) = c_{ij}|u| + O(1),\ K_{33}^*(\alpha,\beta) = c_{33}|u| + O(1)\\ &K_{i3}^*(\alpha,\beta) = K_{3j}^*(\alpha,\beta) = c_{ij}|u|^3 e^{-c|u|}(1+O(|u|^{-1}))\\ &c>0,\ i,j=1,2\\ &M_{ij}^*(\alpha,\beta) = d_{ij} + O(|u|^{-1}),\quad i,j=1,2,3\end{aligned} \tag{1.5}$$

where c_{ij} and d_{ij} are some constants.

We introduce the class $G_\alpha\ (\alpha>1)$ of functions $f(x,y)$ that vanish at the boundary of region $\Omega\ (x^2+y^2=1)$ and whose derivatives with respect to each variable belong to $L_\alpha\ (\alpha>1)$. This class of functions is imbedded in the space of functions for which the integral

$$\int_{-\infty}^{\infty}\!\!\int \sqrt{\alpha^2+\beta^2+k^2}\,|Q(\alpha,\beta)|^2\,d\alpha\,d\beta < \infty \tag{1.6}$$

where k is some number, is convergent.

We denote by $\pm z_n\ (n=1,2,\ldots,N)$ the poles of functions $S,\ T,\ L,\ K$ (the same for all elements). The theorem of uniqueness holds for the system of integral equations (1.1).

Theorem 1. If region Ω is convex, the system of integral equations (1.1) cannot have in $G_\alpha\ (\alpha>1)$ more than one solution, when S, T, L, K satisfy the conditions:

1°. $[S^{-1}(z_n)]' > 0,\ [T^{-1}(z_n)]' > 0,\ n=1,2,\ldots,N.$
2°. $[K^{-1}(z_n)]'\,[S^{-1}(z_n)]' - \{[L^{-1}(z_n)]'\}^2 > 0,\quad n=1,2,\ldots,N.$

3°. There exists a matrix $\mathbf{\Pi}(u)$, $u^2=\alpha^2+\beta^2$ with elements $\Pi_{mn}(u)$ that are rational functions bounded at infinity, with poles at points $\pm z_n$ such that for any $u\ (-\infty<u<\infty)$ the real Hermitian component of matrix $\mathbf{K}^*(\alpha,\beta)\,\mathbf{\Pi}^{-1}(u)$ is positive definite.

Proof of this theorem is analogous to that described in /4,5/ and is omitted here.

2. Let us continue the right-hand sides of equations of system (1.1) over the whole plane of vector functions $\psi(x,y)$, and pass to a cylindrical coordinate system followed by expansion of all functions in the Fourier–Bessel series

$$f(r,\varphi) = \sum_{m=-\infty}^{\infty} f_1(r,m)\,e^{im\varphi} \tag{2.1}$$

Application of the integral transform to the extended system of three Eqs. (1.1) reduces its solution, after linear transformation, to solving for each integral m the system of two equations

$$\mathbf{K}(u)\,\mathbf{U}(u,m) = \mathbf{F}(u,m) + \mathbf{\Psi}(u,m) \tag{2.2}$$

where vector $\mathbf{F}(u,m)$ and the elements of second order matrices $\mathbf{K}(u)$ and $\mathbf{M}(u)$ are of the form

$$\mathbf{F}(u,m)=\frac{a}{G}\left[\mathbf{T}(u,m)+\mathbf{M}(u)\,\mathbf{Q}(u,m)\right] \tag{2.3}$$

$K_{11}(u)=u^2S,\ K_{12}(u)=K_{21}(u)=uL,\ \ K_{22}(u)=K,\ \ M_{11}(u)=u^2s,\ \ M_{12}(u)=-ul,\ \ M_{21}(u)=ul_1,\ \ M_{22}(u)=k$

and one separate equation which is solved independently of (2.2)

$$K_{33}(u)\,U_3(u,m)=F_3(u,m)+\Psi_3(u,m) \tag{2.4}$$

where

$$F_3(u,m)=\frac{a}{G}\left[T_3(u,m)+M_{33}(u)\,Q_3(u,m)\right], \qquad K_{33}(u)=u^2T,\ M_{33}(u)=u^2t \tag{2.5}$$

Components of the two-dimensional vector $\mathbf{U}(u,m)$ and function $U_3(u,m)$ are of the form

$$U_1(u,m)=\int_0^1\left[(u_r^*(r,m)-iu_\varphi^*(r,m))\,J_{m-1}(ru)-(u_r^*(r,m)+iu_\varphi^*(r,m))\,J_{m+1}(ru)\right]r\,dr \tag{2.6}$$

$$U_2(u,m)=2\int_0^1 u_z^*(r,m)\,J_m(ru)\,r\,dr$$

$$U_3(u,m)=\int_0^1\left[(iu_r^*(r,m)+u_\varphi^*(r,m))\,J_{m-1}(ru)+(iu_r^*(r,m)-u_\varphi^*(r,m))\,J_{m+1}(ru)\right]r\,dr$$

Here and subsequently $J_n(x)$ and $H_n^{(2)}(x)$ are Bessel functions of the first and third kind, respectively.

The components of vectors $\mathbf{T}(u,m)$ and $\mathbf{Q}(u,m)$ and of functions $T_3(u,m)$ and $Q_3(u,m)$ are of similar form. The components of vector $\Psi(u,m)$ and function $\Psi_3(u,m)$ which are integral transforms of $\psi(r,m)$ are also determined by formulas (2.6), but their integrals have to be taken over the interval 1 to ∞.

It is convenient to consider Eq. (2.4) as a particular case of the system of Eqs. (2.2) with a first order matrix and solve it similarly to (2.2).

3. Below we use the method of left-side regularization of the system of Eqs. (2.2), which is effective when the conditions of the problem necessitate the determination of the continuation of the right-hand side of (2.2), i.e. of the vector function that defines the behavior of the system of integral equations outside the specified region. The system of Eqs. (2.2) is then directly solved using quadratures of the extended right-hand side. Application of this method and of the left-side regularization to the system of integral equations (2.2) was validated in /5/.

The indicated method and the factorization of matrix functions followed by the projection of the system of Eqs. (2.2) on the upper and lower half-planes of the complex variable u yield for the continuation of the right-hand sides of equations of this system an expression in terms of the two-dimensional vector function $\mathbf{Y}^+(r,m)$ which is regular in the upper half-plane and is determined by the uniquely solvable system of integral equations of the second kind

$$\mathbf{Y}^+(r,m)+\mathbf{N}\mathbf{Y}^+(r,m)=\mathbf{D}(r,m) \tag{3.1}$$

$$\mathbf{N}\mathbf{Y}^+(r,m)=\frac{1}{4\pi^2}\int_{\Gamma_1}\frac{\mathbf{K}_+^{-1}(\alpha)}{\alpha-r}\,d\alpha\int_{\Gamma_-}\frac{\mathbf{\Theta}(\rho,\alpha,m)}{\rho^2-\alpha^2}\,\mathbf{K}_+(\rho)\,\mathbf{Y}^+(\rho,m)\,d\rho$$

$$\mathbf{D}(r,m)=\frac{1}{2\pi i}\int_{\Gamma_1}\frac{\mathbf{K}_+^{-1}(\alpha)}{\alpha-r}\,\mathbf{Z}(\alpha,m)\,H_m^{(2)}(\alpha)\,J_m(\alpha)\,\alpha\,d\alpha$$

$$Z_1(\alpha,m)=\frac{1}{2}\int_{\Gamma}\frac{\Theta_{11}^*(\rho,\alpha,m)}{\rho^2-\alpha^2}\,F_1(\rho,m)\,H_m^{(2)}(\rho)\,\rho\,d\rho, \qquad Z_2(\alpha,m)=\frac{F_2(\alpha,m)}{J_m(\alpha)}$$

where $\mathbf{K}_+(u)$ and $\mathbf{K}_-(u)$ are factors resulting from the left-side factorization of matrix function $\mathbf{K}(u)$ with respect to contour Γ /5,6/

$$\mathbf{K}(u)=\mathbf{K}_+(u)\,\mathbf{K}_-(u) \tag{3.2}$$

and $K_+(u)$ and $\mathbf{K}_-(u)$ are matrices whose elements are regular and have no zeros in the upper and lower half-planes, respectively. Contour Γ_1 coincides everywhere with the real axis, bypassing the real positive and negative poles and zeros of $\mathbf{K}(u)$, respectively, from below and above. Contour Γ_- lies below Γ_1 but the integrand between Γ_1 and Γ_- is regular. Point r lies above Γ_1.

The solution of (2.4) similarly reduces to solving an equation of the form (3.1) in which functions $\mathbf{K}_+(u)$ and $\mathbf{K}_-(u)$— the results of factorization of $K_{33}(u)$ with respect to contour Γ—are substituted for matrices $\mathbf{K}_+(u)$ and $K_-(u)$, the elements $\Theta_{11}(\rho,\alpha,m)$ of matrix $\mathbf{\Theta}(\rho,\alpha,m)$ substituted for the matrix itself, and function

$$Z_3(\alpha, m) = \frac{1}{2}\int_\Gamma \frac{\Theta_{11}^*(\rho, \alpha, m)}{\rho^2 - \alpha^2} F_3(\rho, m) H_m^{(2)}(\rho)\rho\, d\rho$$

substituted for vector $\mathbf{Z}(\alpha, m)$.

System (3.1) can be reduced to a Fredholm system, and its solution then derived by means of external analysis /7/. However, the method based on the approximate reduction to a system of linear algebraic equations /8/ with approximate factorization of the matrix function $\mathbf{K}(u)$ is more convenient. Splitting the operator into a finite-dimensional and a small one which can be neglected on the basis of available estimates /5/ (the presence in it of decreasing exponents) is obtained by deforming contours Γ_1 and Γ_- downward to the branching point $u = -ib$ introduced in matrix $\mathbf{K}(u)$ in the course of approximation with the addition of integrand residues taken at poles of that function at intersections with deformed contours.

We express the components of vector $\mathbf{\Psi}(u, m)$ in terms of $\mathbf{Y}^+(u, m)$ and, after inversion of formulas (2.6), we obtain for the Fourier coefficients of vector function $\boldsymbol{\psi}(r, m)$ in a cylindrical coordinate system expressions of the form

$$\psi_r(r, m) = \frac{m}{4r}\int_\Gamma [\Psi_1(\alpha, m) + i\Psi_3(\alpha, m)]\frac{H_m^{(2)}(\alpha r)}{H_m^{(2)}(\alpha)}d\alpha - \frac{1}{4}\int_\Gamma \Psi_1(\alpha, m)\frac{H_{m-1}^{(2)}(\alpha r)}{H_m^{(2)}(\alpha)}\alpha\, d\alpha \tag{3.3}$$

$$\psi_\varphi(r, m) = \frac{m}{4r}\int_\Gamma [\Psi_3(\alpha, m) - i\Psi_1(\alpha, m)]\frac{H_m^{(2)}(\alpha r)}{H_m^{(2)}(\alpha)}d\alpha - \frac{1}{4}\int_\Gamma \Psi_3(\alpha, m)\frac{H_{m-1}^{(2)}(\alpha r)}{H_m^{(2)}(\alpha)}\alpha\, d\alpha$$

$$\psi_z(r, m) = -\frac{1}{4}\int_\Gamma \Psi_2(\alpha, m)\frac{H_m^{(2)}(\alpha r)}{H_m^{(2)}(\alpha)}\alpha\, d\alpha$$

$$\mathbf{\Psi}(\alpha, m) = \alpha^{-1}\mathbf{K}_+(\alpha)\mathbf{Y}^+(\alpha, m) + \mathbf{F}(\alpha) H_m^2(\alpha), \quad \Psi_3(\alpha, m) = \alpha^{-1}K_+(\alpha) Y_3^+(\alpha, m) + F_3(\alpha) H_m^{(2)}(\alpha) \tag{3.4}$$

Contours Γ is above contour Γ_- and $\operatorname{Im}\alpha < \operatorname{Im}\rho$. Elements of the diagonal matrices $\mathbf{\Theta}(\rho, \alpha, m)$ and $\mathbf{\Theta}^*(\rho, \alpha, m)$ are of the form

$$\Theta_{11}^*(\rho, \alpha, m) = \rho\frac{J_{m+1}(\alpha)}{J_m(\alpha)} - \alpha\frac{H_{m+1}^{(2)}(\rho)}{H_m^{(2)}(\rho)}, \quad \Theta_{22}^*(\rho, \alpha, m) = \alpha\frac{J_{m+1}(\alpha)}{J_m(\alpha)} - \rho\frac{H_{m+1}^{(2)}(\rho)}{H_m^{(2)}(\rho)} \tag{3.5}$$

$$\Theta_{jj}(\rho, \alpha, m) = -\pi\alpha i H_m^{(2)}(\alpha) J_m(\alpha)\Theta_{jj}^*(\rho, \alpha, m) + (\rho + \alpha), j = 1, 2$$

After the vector function $\boldsymbol{\psi}(x, y)$—the continuation of the right-hand sides of equations of system (1.1) outside of the specified region Ω—has been determined using formulas (2.1) and (3.3), the solution (the vector function $\mathbf{u}^*(x, y)$) of that system can be obtained by a double application of integral transforms to system (1.1) whose right-hand sides are now specified over the whole plane.

4. The following asymptotic estimates can be obtained for the vector function $\mathbf{Y}^+(\rho, m)$ and function $Y_3^+(\rho, m)$ as $|\rho| \to \infty$.

$$\mathbf{Y}^+(p, m) = \mathbf{c}_1\rho^{-1} + O(\rho^{-2}), \quad Y_3^+(\rho, m) = c_2\rho^{-1} + O(\rho^{-2}) \tag{4.1}$$

where $\mathbf{c}_1$ is a constant vector and $c_2 = \text{const}$).

The expression for the stress vector $\boldsymbol{\tau}(x, y)$ along the continuation of the crack upper edge, i.e. for $x^2 + y^2 > 1$, is readily obtained in terms of the extension of the right-hand side of $\boldsymbol{\psi}(x, y)$ of the system of Eqs. (1.1)

$$\boldsymbol{\tau}(x, y) = \frac{G}{a}\boldsymbol{\psi}(x, y) - \frac{1}{4\pi^2}\int_{\Gamma_1}\int_{\Gamma_2}\mathbf{M}^*(\alpha, \beta)\,\mathbf{Q}^*(\alpha, \beta)\, e^{-i(\alpha x + \beta y)} d\alpha\, d\beta, \quad x^2 + y^2 > 1 \tag{4.2}$$

Using estimates (4.1) for components of the vector function $\boldsymbol{\tau}(x, y)$ we obtain the asymptotic formulas

$$\tau_i(x, y) = \frac{A_i}{\sqrt{r - 1}}, \quad r = (x^2 + y^2)^{1/2} \to 1, \quad r > 1, \quad i = 1, 2, 3 \tag{4.3}$$

where A_i are some constants proportional to dynamic intensity coefficients of respective stresses at the crack tip. Application of the method of left side regularization for solving the system of integral equations enables us to determine immediately the dynamic stress intensity coefficient, an important parameter in fracture mechanics, without having to solve the system itself.

5. In the numerical solution of the problem we are faced with the problem of factorization of the matrix function $\mathbf{K}(u)$ of the form (3.2). Elements of that matrix have the following

properties:

1° they are regular along the whole real axis, except at a finite number of points (poles) that are the same for all elements;

2° functions $K_{11}(u)$ and $K_{22}(u)$ are even, while $K_{12}(u) = K_{21}(u)$ are odd, and

3° as $u \to \infty$ the matrix elements are of the following order:

$$K_{11}(u),\ K_{22}(u) = O(|u|),\quad K_{12}(u),\ K_{21}(u) = O(e^{-c|u|})$$

Owing to the extreme complexity of the elements of matrix $\mathbf{K}(u)$ which have such properties, its factorization can be effected only approximately. For this matrix $\mathbf{K}(u)$ is approximated by matrix $\mathbf{H}(u)$ with rational fractional elements multiplied by $(u^2+b^2)^{1/2}$, where b is some constant. Validity of this approximation is substantiated by the following theorem.

Theorem 2. Let $\mathbf{u}^{(1)}(x, y) \equiv \{u_1^{(1)}(x, y),\ u_2^{(1)}(x, y)\}$ and $\mathbf{u}^{(2)}(x, y) \equiv \{(u_1^{(2)}(x, y), u_2^{(2)}(x, y)\}$ be solutions of Eq. (1.1) when $\mathbf{K}(\alpha, \beta) = \mathbf{K}^{(1)}(\alpha, \beta)$ and $\mathbf{K}(\alpha, \beta) = \mathbf{K}^{(2)}(\alpha, \beta)$ respectively, with equal right-hand sides. Then, when the conditions of Theorem 1 and the conditions

$$|K_{ij}^{(1)}(\alpha, \beta) - \mathbf{K}_{ij}^{(2)}(\alpha, \beta)|(\det \mathbf{K}^{(1)}(\alpha, \beta))^{-1}(1 + (\alpha^2+\beta^2)^{1/2})^{\gamma} < \varepsilon,\ \gamma > 1/2$$

are satisfied for fairly small ε, we have

$$|u_k^{(1)}(x, y) - u_k^{(2)}(x, y)| < \delta(\varepsilon),\quad k = 1, 2$$

Proof of this theorem, based on the method of perturbations, is omitted here for brevity.

Matrix $\mathbf{H}(u)$ can be represented in the form

$$\mathbf{H}(u) = (u^2+b^2)^{1/2} \prod_{k=1}^{N} (u^2 - z_k^2)^{-1} \mathbf{P}(u) \tag{5.1}$$

where z_k $(k = 1, 2, \ldots, n\ (n < N))$ are real poles of elements of matrix $\mathbf{K}(u)$, z_k $(k = n+1, \ldots, N)$ are complex numbers, zeros of the denominator of rational fractional functions that approximate the elements of matrix $\mathbf{K}(u)$ and are the same for all elements; $\mathbf{P}(u)$ is a polynomial matrix whose elements $P_{mm}(u)$ are polynomials of degree $2N$ and $P_{mn}(u)$ $(m \neq n)$ polynomials of degree $2N - R$ $(R > 0)$. It is clear from (5.1) that matrix $|u|^{-1}\mathbf{H}(u)$ degenerates to a unit matrix as $|u| \to \infty$. Matrix $\mathbf{P}(u)$ can be represented in the form /9/

$$\mathbf{P}(u) = \mathbf{C}(u)\,\mathbf{G}(u)\,\mathbf{D}(u) \tag{5.2}$$

where $\mathbf{G}(u)$ is a diagonal matrix with elements $G_{11}(u) = \mathrm{const}$, and $G_{22}(u)$ is a polynomial of degree $4N$; $\mathbf{D}(u)$ and $\mathbf{C}(u)$ are matrices with regular elements and constant determinants.

When the zeros of element $G_{22}(u)$ are known, matrix $\mathbf{G}(u)$ can be readily represented in the form $\mathbf{G}(u) = \mathbf{G}_+(u)\,\mathbf{G}_-(u)$. The formula

$$\mathbf{C}(u)\,\mathbf{G}_+(u) = \mathbf{G}_+^{\circ}(u)\,\mathbf{B}(u) \tag{5.3}$$

is derived similarly to (5.2). The properties of matrix $\mathbf{B}(u)$ are the same as of matrices $\mathbf{C}(u)$ and $\mathbf{D}(u)$. This additional transform is required if the matrices

$$\mathbf{K}_+(u) = (u+ib)^{1/2} \prod_{k=1}^{N} (u+z_k)^{-1}\mathbf{G}_+^{\circ}(u),\quad \mathbf{K}_-(u) = (u-ib)^{1/2} \prod_{k=1}^{N} (u-z_k)^{-1}\mathbf{B}(u)\,\mathbf{G}_-(u)\,\mathbf{D}(u) \tag{5.4}$$

obtained by the factorization of $\mathbf{H}(u)$, are to satisfy not only (3.2) but, also, if matrices $1/\sqrt{u}\,\mathbf{K}_+(u)$ and $1/\sqrt{u}\mathbf{K}_-(u)$ are to degenerate in unit matrices as $|u| \to \infty$.

6. As an example of solution derivation by the proposed method, let us consider the solution of the system of integral equations (1.1) with matrix $\mathbf{K}^*(\alpha, \beta)$ and its elements

$$K_{jj}^*(\alpha, \beta) = \frac{u^2+2t^2}{u^2-2it^2},\quad u^2 = \alpha^2+\beta^2,\quad j = 1, 2, 3$$

$$K_{31}^*(\alpha, \beta) = -K_{13}^*(\alpha, \beta) = i\alpha\frac{2t}{u^2-2it^2},\quad K_{12}^*(\alpha, \beta) = K_{21}^*(\alpha, \beta) = 0$$

$$K_{32}^*(\alpha, \beta) = -K_{23}^*(\alpha, \beta) = i\beta\frac{2t}{u^2-2it^2}$$

and right-hand side $f_x(x, y) = f_y(x, y) = 0,\ f_z(x, y) = 1$.

Solution of that system of equations reduces to the solution of the system of Eqs. (2.2) and of the single equation (2.4) only for $m = 0$. In our problem the functions and matrix elements are

$$K_{jj}(u) = \frac{u^2+2t^2}{u^2-2it^2},\quad j = 1, 2, 3;\quad K_{12}(u) = K_{21}(u) = \frac{2tu}{u^2-2it^2}$$

The factorization of matrix $\mathbf{K}(u)$ yields for the elements of matrices $\mathbf{K}_+(u)$ and $\mathbf{K}_-(u)$ the expressions

$$K_{11}^{\pm}(u) = K_{22}^{\pm}(u) = (u \pm it)(u \pm t(i+1))^{-1}, \quad K_{12}^{\pm}(u) = K_{21}^{\pm}(u) = t(u \pm t(i+1))^{-1}$$

Solving the algebraic system to which system (3.1) reduces, we obtain

$$Y_1^+(-t(i+1)) = -Y_2^+(-t(i+1)) = \frac{4J_1(-t(i-1))}{\pi(i-1)J_0(-t(i-1))\,\Theta^\circ(-t(i+1),\,-t(i-1))}$$

$$\Theta^\circ(\alpha,\beta) = (\alpha+\beta)(H_0^{(2)}(\alpha)J_1(\beta) - H_1^{(2)}(\alpha)J_0(\beta))$$

and for the single equation we have $Y_3^+(-t(i+1)) \equiv 0$.

Components of the vector function $\psi(x, y)$ — continuation of the right-hand sides of the system of integral equations (1.1) outside the specified region of the cylindrical coordinate system — are of the form

$$\psi_r(r,0) = \frac{4tJ_1(-t(i-1))}{(i+1)\Theta^\circ(-t(i+1),\,-t(i-1))} H_1^{(2)}(-rt(i+1)) \tag{6.1}$$
$$\psi_\varphi(r,0) = 0$$
$$\psi_z(r,0) = \frac{4tJ_1(-t(i-1))}{(i+1)\Theta^\circ(-t(i+1),\,-t(i-1))} H_0^{(2)}(-rt(i+1))$$

and the solution itself is, respectively,

$$u_r^*(r,0) = -\frac{4tH_1^{(2)}(-t(i+1))}{(i+1)\Theta^\circ(-t(i+1),\,-t(i-1))} J_1(-rt(i-1)) \tag{6.2}$$
$$u_\varphi^*(r,0) = 0$$
$$u_z^*(r,0) = -i - \frac{4tH_1^{(2)}(-t(i+1))}{(i+1)\Theta^\circ(-t(i+1),\,-t(i-1))} J_0(-rt(i-1))$$

whose substitution into the input system (1.1) yields an identity, which proves the correctness of (6.2).

7. A set of applied programs compiled at the Scientific Research Institute of Applied Mathematics of the Rostov State University makes possible the approximation and factorization of matrices, and the calculation of characteristics of a crack with edges subjected to harmonic oscillations. These programs were used for determining the dynamic stress intensity coefficient at the crack tip under the application of normal oscillating load to its edges

$$\tau(x, y) \equiv \{0, 0, \sigma\}, \quad \sigma = \text{const}, \quad x, y \in \Omega$$

In Fig.1 is shown the dependence of the dimensionless quantity $|K/\sigma|$ on parameter $\varkappa_2$ (see (1.4)), where K is a dynamic coefficient of tension intensity in the top crack. The solid line relates to crack located at distance 0.2 of its thickness, but dash and dash-dot lines relate, respectively, to cracks located at distances 0.25 and 0.3 of crack thickness.

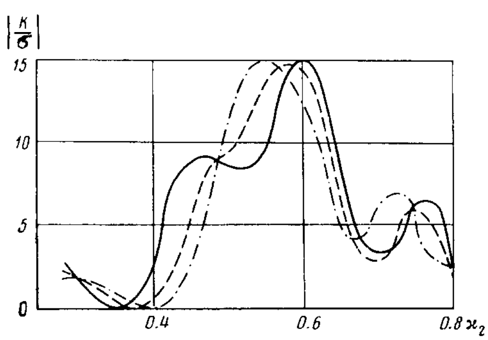

Fig.1

The dependence of $|K/\sigma|$ relation is on the frequency of the crack banks oscillation and on it layout in the layer.

The behavior of these curves substantially differs from that shown in /10/, where the plane problem of crack edge oscillation in an elastic space was investigated. Their comparison clearly shows that the scale factor (a crack of finite dimensions in our problem and an infinite one in one direction in the plane crack problem) has an appreciable effect on the dynamic stress intensity coefficient. The presence of some frequencies $\varkappa_2^{(i)}$, $i = 1, 2, \ldots$, at which that coefficient is close to zero should be noted.

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